Consequently

$$\left(\frac{r-s}{\alpha}-\|x\|\right)\frac{1}{\varepsilon-\|x\|}>\lambda(z)$$

Using the inequality $(m(z), x)\leqslant\|m(z)\|\|x\|=\varepsilon\|x\|$, we have

$$(-m(z)/\varepsilon,\ x+\lambda(z)\ (m(z)-x))>(s-r)/\alpha$$

Using Eq.(2.3) we obtain the required condition (3.8).

The game can, thus be completed from the initial position z° in a finite time, if its parameters are connected by relation (3.9). The controls of the pursuers are constructed as in the preceding example.

REFERENCES

1. PSHENICHNYI B.N., Simple pursuit by several objects. Kibernetika, 3, 1976.
2. PSHENICHNYI B.N. and RAPPOPORT I.S., On the problem of group pursuit. Kibernetika, 6, 1979.
3. PSHENICHNYI B.N., CHIKRII A.A. and RAPPOPORT I.S., An effective method of solving differential games with many pursuers. Dokl. AN SSSR, 256, 3, 1981.
4. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games. Moscow, Nauka, 1974.
5. PONTRYAGIN L.S., Linear differential games of pursuit. Matem. sb., 112, Issue 3, 1980.
6. GRIGORENKO N.L., On the linear problem of pursuit by several objects. Dokl. AN SSSR, 258, 2, 1981.
7. PSHENICHNYI B.N., Convex Analysis Extremal Problems. Moscow, Nauka, 1980.
8. PSHENICHNYI B.N., The Necessary Conditions for an Extremum. Moscow, Nauka, 1982.
9. VARGA J., Optimal Control by Differential and Functional Equations. Moscow, Nauka, 1977.

Translated by J.J.D.

PMM U.S.S.R.,Vol.49,No.4,pp.404-413,1985 0021-8929/85 $10.00+0.00
Printed in Great Britain Pergamon Journals Ltd.

OPTIMAL CONTROL WITH A FUNCTIONAL AVERAGED ALONG THE TRAJECTORY*

A.I. PANASYUK and V.I. PANASYUK

A set of infinite optimal trajectories (IOT) is defined. It is shown that in an arbitrary fixed time interval any optimal trajectory of a system for a problem with fairly large control time (and arbitrary initial conditions) can be uniformly approximated to some IOT with the desired accuracy. Sufficient conditions are presented which ensure the existence of IOT, and the structure of the IOT set is investigated, using the rearrangement operator. The set of main trajectories is defined, and the correctness of that definition is proved. A chain of approximations is obtained: IOT approximate optimal trajectories of finite length, and the main trajectories approximate the IOT.

The properties of optimal trajectories of considerable length, and of IOT and main trajectories are investigated by solving the problem of optimal control, with a functional averaged along the trajectory. It is shown that a limit time-averaged value of the quality functional on optimal trajectories of the problems in a finite interval, when its duration increases without limit, does exist, is independent of the selection of the initial and finite conditions of these problems, and is equal to its value on any IOT. For a problem of "optimum in the mean" control the exact lower bound of the functional averaged over time does not change, if one limits the consideration only to periodic modes of the system with all possible periods. The paper continues investigations carried out in /1–4/. A somewhat different aspect of the problem of the asymptotic forms of the optimal trajectories of a control system was studied in /5, 6/, and a number of similar problems was investigated in /7–11/ etc. Generalizations to problems with discrete times were considered in /12, 13/.

1. Formulation of the problem. The following problem of optimal control is considered:

$$\frac{dx}{dt}=f(x,\ u),\quad u\in U\subset R^r;\quad x\in X\subset R^n \tag{1.1}$$

Prikl.Matem.Mekhan.,49,4,524-535,1985

$$I(x(\cdot),\ u(\cdot),\ t_0,\ T)=\int_{t_0}^{T} F(x,\ u)\,dt\rightarrow \min;\quad t_0,\ T=\mathrm{const} \tag{1.2}$$

where the set X is closed and U is compact. The functions $f(x, u)$, $F(x, u)$ are continuous, and $f(x, u)$, in addition, satisfies the following condition: $L>0$ and $\varepsilon>0$ are obtained for $x\in X$, such that $\|f(x', u)-f(x'', u)\|\leqslant L\|x'-x''\|$, when $x', x''\in X$, $\|x'-x\|<\varepsilon$, $\|x''-x\|<\varepsilon$, $u\in U$. Here $\|\cdot\|$ is the Euclidean norm. The measurable vector functions $u(t)\in U$ represent admissible controls, and the absolutely continuous vector functions $x(t)$ that satisfy almost everywhere (1.1) for some admissible control represent the trajectories. The asymptotic properties of optimal trajectories are investigated as $T\rightarrow\infty$.

2. Infinite optimal trajectories. The admissible control $u^\circ(t)$, $t_0\leqslant t\leqslant T$ and any of its respective trajectories $x^\circ(t)$, $t_0\leqslant t\leqslant T$ are called optimal, if for any other admissible control $u(t)$ and any corresponding trajectory $x(t)$, $t_0\leqslant t\leqslant T$ that satifies the boundary conditions $x(t_0)=x^\circ(t_0)$, $x(T)=x^\circ(T)$, the inequality $I(x(\cdot), u(\cdot), t_0, T)\geqslant I(x^\circ(\cdot), u^\circ(\cdot), t_0, T)$ is satisfied. The principle of optimality consists in the fact that any part (arc) $x^\circ(t)$, $t_0\leqslant \xi_1\leqslant t\leqslant\xi_2\leqslant T$ of the optimal trajectory $x^\circ(t)$ is itself an optimal trajectory.

Definition. An admissible control $u^\circ(t)$ and some corresponding trajectory $x^\circ(t)$ defined on the set J from R of one of the following types: $-\infty<t<\infty$; $-\infty<t\leqslant b$; $a\leqslant t\leqslant b$; $a\leqslant t<\infty$ are called optimal, if for any segment $[\xi_1, \xi_2]\subset J$ the contractions $x^\circ(t)$, $u^\circ(t)$ on $\xi_1\leqslant t\leqslant\xi_2$ are the optimal trajectories and control. We call the optimal trajectory $x^\circ(t)$, $-\infty<t<\infty$ an infinite optimal trajectory (IOT). /2, 4/.

3. Passing to the limit over successions of trajectories. We use the following notation:

$$q^*=(q,\ q_{n+1})\in R^{n+1}$$
$$G^*(x)=\{q^*:\ q=f(x,u),\ q_{n+1}\geqslant F(x,u),\ u\in U\}$$

Theorem 3.1. For $x\in X$ suppose the set $G^*(x)$ is convex and the succession of trajectories $x^k(t)$, $-\infty<a\leqslant t\leqslant b<\infty$, $k\rightarrow\infty$ corresponding to some admissible $u^k(t)$ is uniformly convergent to $x(t)$ on $[a, b]$. Then $x(t)$ is the trajectory that corresponds to some admissible control $u(t)$, $a\leqslant t\leqslant b$ that satisfies the inequality

$$I(x(\cdot),\ u(\cdot),\ a,\ b)\leqslant \varliminf_{k\rightarrow\infty} I(x^k(\cdot),\ u^k(\cdot),\ a,\ b) \tag{3.1}$$

The proof is carried out using the scheme given in /14/ pp.95-104.

We denote by $A(\varepsilon, x)$, $\varepsilon>0$ the set of points in X reached from $x\in X$ in exactly the time ε. Correspondingly, $A(-\varepsilon, x)\subset X$ is the set of points from which x is reached in a time ε. We say that system (1.1) is positively (negatively) locally controllable along the trajectory $x(t)$, $a\leqslant t\leqslant b$ when $t=t^*\in(a, b)$, if an $\varepsilon_0>0$ can be found for which when $0<\varepsilon<\varepsilon_0$ the inclusion $x(t^*+\varepsilon)\in \operatorname{Int} A(\varepsilon, x(t^*))$ holds, (respectively $x(t^*-\varepsilon)\in\operatorname{Int} A(-\varepsilon, x(t^*))$). We say that the system is controllable in the limit on an infinite trajectory $x(t)$, $-\infty<t<\infty$, if successions $t_k'\rightarrow-\infty$, $t_k''\rightarrow\infty$ as $k\rightarrow\infty$ are found such that the system is negatively locally controllable on $x(\cdot)$ when $t=t_k'$, and positively locally controllable along $x(\cdot)$ when $t=t_k''$, $k\geqslant 1$.

From (3.1) and the limit controllability we obtain.

Theorem 3.2. Let the set $G^*(x)$ be convex when $x\in X$, and the continuous function $x(t)\in X$, $-\infty<t<\infty$ be on each segment $a\leqslant t\leqslant b$ is a uniform limit of some succession of optimal trajectories, dependent on $[a, b]$. Then $x(t)$ is a trajectory. If the system (1.1) is additionally controllable on $x(\cdot)$, then $x(\cdot)$ is an IOT.

4. The rearrangement operator Π /2, 4/. We denote for the compactum $D\subset X$ the topological space $C(R, D)$ of all continuous mappings $R\rightarrow D$ with topology of uniform convergence on compacta. For the subset $W'\subset C(R, D)$ we define the rearrangement operator $\Pi(W')\subset C(R, D)$ transforming the subsets from $C(R, D)$ into subsets from $C(R, D)$ according to the formulae

$$\Pi(W')=\overline{SW'},\quad SW'=\bigcup_{\varphi(\cdot)\in W'}\ \bigcup_{\tau\in R}\varphi_\tau(\cdot);\quad \varphi_\tau(t)\equiv\varphi(t+\tau)$$

and the closure is taken in $C(R, D)$, i.e. the operator Π converts W' into the subset from $C(R, D)$ obtained by the closure of all possible displacement $\varphi_\tau(\cdot)$ of all mappings from W'. The properties of the rearrangement operator can be verified directly.

Theorem 4.1. Let W', $W''\subset C(R, D)$. Then $S\Pi(W')=\Pi(W')$; $W'\subset\Pi(W')$; $\Pi\Pi(W')=\Pi(W')$; $\Pi(W')\cup\Pi(W'')=\Pi(W'\cup W'')$; $\Pi(W'\cap W'')\subset\Pi(W')\cap\Pi(W'')$.

Consider some subset $V \subset C(R, D)$ invariant under the rearrangement operator $\Pi(V) = V$, and introduce on V a new topology, assuming to be closed those and only those subsets $V' \subset V$ that $\Pi(V') = V'$. We denote the topological space obtained by V_*. It follows from Theorem 4.1 that Π is an operator of closure. Then by virtue of the Kuratowski theorem /15/ we obtain that the sets invariant relative to the operator Π can be investigated using topological means.

Corollary 4.1. The topology of the space V_* is correctly defined.

Let $W' \subset C(R, D)$ be some set of IOT. We say that the optimality is invariant under the action of Π on W', when $\Pi(W')$ consists of IOT. We denote by W_D the set of all IOT lying in D, From Theorem 3.2 we obtain the sufficient conditions of invariance of optimality relative to Π.

Corollary 4.2. Let the set $G^*(x)$ be convex, and when $x \in D$ the system is controllable in the limit on any trajectory $x(t) \in D$, $-\infty < t < \infty$ lying in the compactum D. Then for any non-empty set of IOT $W' \subset C(R, D)$, $W' \neq \varnothing$, the optimality is invariant under the action of Π on W'. If in addition $W_D \neq \varnothing$, then $\Pi(W_D) = W_D$.

5. The set of main trajectories.

For the compactum $D \subset X$ the set $W_D^c \subset W_D$, which is non-empty and satisfies three conditions:

1) of approximation: $\Pi(\varphi(\cdot)) \cap W_D^c \neq \varnothing$ when $\varphi(\cdot) \in W_D$,

2) of closure $\Pi(W_D^c) = W_D^c$.

3) of minimality: W_D^c does not contain a proper subset that does satisfy the conditions of approximation and closure, will be called the set of main trajectories for W_D. The correctness of this definition is confirmed by the following theorem.

Theorem 5.1. Let the optimality $W_D \neq \varnothing$ be invariant under the action of Π of W_D. Then $\Pi(W_D) = W_D$: the set of main trajectories W_D^c for W_D exists and is unique.

Proof. Consider the set Φ of all subsets $\varnothing \neq W' \subset W_D$ that satisfy the conditions of approximation and closure. Then $W_D \in \Phi$, and it can be shown that the intersection of a finite number of subsets from Φ is, again, an element of Φ, i.e. Φ has the property of finite intersection /15/. The equipotential continuity trajectories in the compactum D, and the invariance $\Pi(W_D) = W_D$ implies, by Ascoli's theorem the compactness of W_D. From this $\cap W' \neq \varnothing$, $W' \in \Phi$ and it is sufficient to set $W_D^c = \cap W'$, $W' \in \Phi$.

If X is a compactum, we use the notation $W = W_X$, $W^c = W_X^c$, and simply call W^c the set of main trajectories.

Theorem 5.2. Let X be a compactum, and let $G^*(x)$ be convex for $x \in X$; for any $T > 0$ a trajectory of duration T, can be found and the system is, in the limit, controllable on any trajectory $x(t) \in X$, $-\infty < t < \infty$. Then the set W of IOT is non-empty: $W \neq \varnothing$ and the optimality of trajectories is invariant to the action of Π on W: $\Pi(W) = W$. The set W^c of main trajectories exists and is unique.

Proof. The availability of trajectories of any duration ensures the presence of minimizing successions defined on any time intervals. Hence, by virtue of Theorem 3.1 optimal trajectories of arbitrary duration also exist. Using Ascoli's theorem for selecting a succession of optimal trajectories defined on a system of intervals which extends without limit and uniformly on compacta, converging to some curve $x(t) \in X$, $-\infty < t < \infty$, we obtain by Theorem 3.2. that $x(\cdot) \in W$, whence $W \neq \varnothing$. The remaining statements follow from Corollary 4.2 and Theorem 5.1.

6. The chain of approximations.

We say that the set of optimal trajectories from D is closed in the topology of uniform convergence on compacta from $(-\infty, \infty)$, if it follows from that $x(t)$ is an IOT and the vector function $x(t) \in D$, $-\infty < t < \infty$ in each segment $[a, b]$ is the uniform limit of some succession of optimal trajectories defined in $[a, b]$ dependent on $[a, b]$. Theorem 3.2 provides the sufficient conditions of such closure.

Corollary 6.1. Let the set $G^*(x)$ be convex when $x \in X$, and system (1.1) be controllable at the limit on any trajectory $x(t) \in D$, $-\infty < t < \infty$ from the compactum D. Then the set of optimal trajectories from D is closed in the topology of uniform convergence on compacta from $(-\infty, \infty)$.

We denote by $W_D(t_1, t_2)$ the set of all optimal trajectories $x(t) \in D$, $t_1 \leqslant t \leqslant t_2$, and by $W_D(t_1, \theta_1, \theta_2, t_2)$ the set of trajectory contractions from $W_D(t_1; t_2)$ on $[\theta_1, \theta_2] \subset [t_1, t_2]$.

Theorem 6.1. (see /2/ p.61). Let $W_D \neq \varnothing$; the set of optimal trajectories from D is closed in the topology of uniform convergence on compacta from $(-\infty, \infty)$. Then for any $\theta_1 < \theta_2$ and $\varepsilon > 0$ we can indicate T_1 and T_2 such that when $t_1 \leqslant T_1$, $t_2 \geqslant T_2$ for any $x^c(\cdot) \in W_D(t_1, \theta_1, \theta_2, t_2)$ we can find an IOT $\varphi(\cdot) \in W_D$ such that $\| x^c(t) - \varphi(t) \| < \varepsilon$ for $t_1 \leqslant \theta_1 \leqslant t \leqslant \theta_2 \leqslant t_2$.

Taking into account that from the closure of the set of optimal trajectories from D their follows the invariance of the optimality relative to the action Π on W_D, from the condition of approximation in the definition of the set of main trajectories the characteristic of approximation properties of the main trajectories can be similarly obtained.

Theorem 6.2. Let $W_D \neq \emptyset$ and the set of optimal trajectories from D be closed in the topology of uniform convergence on compacta from $(-\infty, \infty)$. Then for any $T > 0$ and $\varepsilon > 0$ an $M = M(T, \varepsilon)$ can be found that satisfies the following condition: for any optimal trajectory $x^0(t) \in D$, $0 \leqslant t \leqslant M$ a main trajectory $\varphi(\cdot) \in W_D^\circ$ and t_1 are found such that $[t_1, t_1 + T] \subset [0, M]$ and $\|x^0(t) - \varphi(t)\| < \varepsilon$ for $t \in [t_1, t_1 + T]$.

Theorems 6.1 and 6.2 show that the sets of W_D and W_D° constitute a chain of approximations: IOT approximate optimal trajectories of finite duration, while the main trajectories reflect the symmetric properties of IOT.

7. Averaging of the functional along the optimal trajectory.

Let us consider now the problems of characterizing IOT and main trajectories, using the problem of optimization with the functional averaged along the trajectory. Let $u_0(t)$ be the optimal control and $x_0(t), t_0 \leqslant t \leqslant t_0 + T$ some optimal trajectory corresponding to it. Then the minimum $I(x_0(\cdot), u_0(\cdot), t_0, t_0 + T) = \min I$ is reached for it in conformity with the definition of an optimal trajectory. The averaged functional is then also minimal

$$\frac{1}{T} I(x_0(\cdot), u_0(\cdot), t_0, t_0 + T) = \min \frac{1}{T} I$$

because T is a given constant. However our aim is not the investigation of one optimization problem for any fixed T, but a complete set of such problems differing by the time T of the process and, also, the clarification of the behaviour of optimal trajectories as $T \to \infty$. Hence letting $T \to \infty$, we obtain the averaged problem of minimizing the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} F(x(t), u(t))\, dt \to \min \tag{7.1}$$

on some set of admissible controls and trajectories on $[t_0, t_0 + T]$. By the same token we take $u(t)$, $x(t)$, $t_0 \leqslant t < \infty$, the limit (7.1) is calculated as $T \to \infty$, and then the minimum of that limit is sought on the set of admissible controls and trajectories on $[t_0, \infty)$.

However, this is insufficient for the statement of the problem of optimal control with a functional averaged along the trajectory to be correct and to be a useful method of investigation. First, the limit (7.1) does not exist for any controls and trajectories, hence the question of its existence must be separately considered. Second, the solution of problem (7.1) must definitely indicate the trajectory on which that minimum is reached. At the same time one and the same value of limit (7.1) as $T \to \infty$, if it exists, corresponds to trajectories and controls in an infinite time interval differing only in some finite time interval. Hence the criterion of optimality (7.1), where a minimum is sought on a fairly wide set of trajectories and controls Ω defines not a single trajectory, but a whole set of trajectories and controls for which a minimum is attained. To avoid such ambiguity one has to narrow the set of pairs Ω on which the minimum (7.1) is sought. Such narrowing may lead to the existence of the limit (7.1) (e.g., if we take $\Omega = \Omega_n$, where Ω_n is the set of admissible periodic modes). Third, the averaged problem (7.1) must have a solution. For instance, by narrowing Ω to periodic modes $\Omega = \Omega_n$, we obtain the problem of periodic optimization (PO) whose solution (optimum cycle), if it exists, is uniquely determined, except the special cases of optimal cycle non-uniqueness. However, the minimum of (7.1) may not be reached. Simultaneously the widening of modes of Ω to the almost periodic modes $\Omega = \Omega_{nn}$ /4/ may ensure the existence of a solution. This shows the value of widening Ω to the set of almost periodic modes.

8. The standard large variation of the trajectory.

We say that the system (1.1) is uniformly controllable on compacta $D \subset X$, if and only if, there exists a compactum $K \subset X$ and a number $M > 0$ such that for any two points $x_0, x_M \in D$ a trajectory $x(t) \in K$, $0 \leqslant t \leqslant M$, $x(0) = x_0$, $x(M) = x_M$ can be found.

Consider two trajectories $x_0(t)$, $x(t) \in D$, $a \leqslant t \leqslant b$, $b - a \geqslant 2M$ of which $x_0(\cdot)$ is optimal. We construct the trajectories $x_1(t) \in K$, $a \leqslant t \leqslant a + M$, $x_1(a) = x_0(a)$, $x_1(a + M) = x(a + M)$, $x_2(t) \in K$, $b - M \leqslant t \leqslant b$, $x_2(b - M) = x(b - M)$, $x_2(b) = x_0(b)$, and determine the larger variation $y(t)$, $a \leqslant t \leqslant b$ of the trajectory $x_0(\cdot)$ by formulae $y(t) = x_1(t)$ when $a \leqslant t \leqslant a + M$; $y(t) = x(t)$ when $a + M \leqslant t \leqslant b - M$, and $y(t) = x_2(t)$ when $b - M \leqslant t \leqslant b$. From the compactness of $K \times U$ and the continuity of $F(x, u)$ it follows that for some $N = N(D)$ the inequalities

$$I(x_1(\cdot), a, a + M) \leqslant N,\ I(x_2(\cdot), b - M, M) \leqslant N$$
$$|I(x(\cdot), a, a + M)| \leqslant N,\ |I(x(\cdot), b - M, M)| \leqslant N$$

hold.

By virtue of the optimality $I(x_0(\cdot), a, b) \leqslant I(y(\cdot), a, b)$, whence we obtain the basic inequality for the standard large variation

$$I(x_0(\cdot), a, b) \leqslant 4N + I(x(\cdot), a, b) \tag{8.1}$$

Let us fix $\omega > 0$. Let there be some sequence $\tau_k \to \infty$, $\tau_k \geqslant \omega$, $k \to \infty$, and functions $g_k(t)$, $t_0 < t < \tau_k$ that are Lebesgue summable. We shall consider the integrals

$$\alpha_k = \frac{1}{\tau_k}\int_{t_0}^{t_0+\tau_k} g_k(t)\,dt, \quad \beta_k(\theta_k) = \frac{1}{\omega}\int_{\theta_k}^{\theta_k+\omega} g_k(t)\,dt$$

Using the Lebesgue integral, we obtain the following statement.

Lemma 8.1. Let $|g_k(t)| < M_0$ when $k \geqslant 1$, $-\infty < t < \infty$ for some M_0. Then θ_k can be selected to satisfy the condition $\beta_k(\theta_k) - \alpha_k \to 0$, $[\theta_k, \theta_k + \omega] \subset [t_0, t_0 + \tau_k]$ as $k \to \infty$.

9. The existence of a unique limit for the functional averaged over optimal trajectories. For brevity, we shall write $I(x(\cdot), a, b)$ instead of $I(x(\cdot), u(\cdot), a, b)$.

Theorem 9.1. Let system (1.1) be uniformly controllable on the compactum $D \subset X$, and for any $T > 0$ suppose the set $W_D(T)$ of optimal trajectories of duration T lying entirely in D is non-empty. Then the limit

$$\lim_{T\to\infty} \frac{1}{T} I(x_T(\cdot), 0, T) = C_0 = C_0(D) \tag{9.1}$$

exists and is independent of the selection of $x_T(\cdot) \in W_D(T)$.

Proof. Assuming that the theorem is false, we find sequences $x_{0k}(\cdot) \in W_D(\tau_k)$ and $x_{0m}(\cdot) \in W_D(T_m)$ that satisfy the inequality

$$C' = \lim_{k\to\infty} \frac{1}{\tau_k}\int_0^{\tau_k} F(x_{0k}(t), u_{0k}(t))\,dt < C'' = \lim_{m\to\infty}\frac{1}{T_m}\int_0^{T_m} F(x_{0m}(t), u_{0m}(t))\,dt \tag{9.2}$$

where $x_{0k}(\cdot), x_{0m}(\cdot)$ correspond to $u_{0k}(\cdot), u_{0m}(\cdot)$, while $\tau_k, T_m \to \infty$ for $k, m \to \infty$. We fix $m \geqslant 1$ such that $T_m \geqslant 2M$ and put $\omega = T_m$. According to Lemma 8.1 θ_k, k_m can be found such that

$$|\beta_k(\theta_k) - \alpha_k| < \frac{1}{m} \quad \text{for} \quad k \geqslant k_m, \quad g_k(t) = F(x_{0k}(t), u_{0k}(t)) \tag{9.3}$$

By an appropriate selection of the reference point on trajectories $x_{0k}(\cdot)$, we can obtain $\theta_k = 0$(then the trajectories themselves are defined for $-\theta_k \leqslant t \leqslant \tau_k - \theta_k$, and $\tau_k - \theta_k \geqslant T_m$). We construct a standard large variation of trajectories, taking $x_{0m}(t)$ as $x_0(t)$, and setting $a = 0$, $b = T_m$. Then, by virtue of (8.1) we obtain

$$I(x_{0m}(\cdot), 0, T_m) \leqslant 4N + I(x_{0k}(\cdot), 0, T_m)$$

which implies

$$T_m^{-1} I(x_{0m}(\cdot), 0, T_m) \leqslant T_m^{-1} I(x_{0k}(\cdot), 0, T_m) + T_m^{-1}4N$$

and from (9.3), taking into account $\theta_k = 0$, we obtain

$$|T_m^{-1} I(x_{0k}(\cdot), 0, T_m) - \tau_k^{-1} I(x_{0k}(\cdot), 0, \tau_k)| \leqslant m^{-1}, \quad k \geqslant k_m$$

It follows from the last two inequalities that

$$T_m^{-1} I(x_{0m}(\cdot), 0, T_m) \leqslant \tau_k^{-1} I(x_{0k}(\cdot), 0, \tau_k) + T_m^{-1}4N + m^{-1}, \quad k \geqslant k_m$$

By letting $m \to \infty$ here we obtain $C'' \leqslant C'$, which contradicts (9.2).

By the problem of average-optimal control, with the functional averaged along the trajectory, we mean the problem of minimizing

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T F(x(t), u(t))\,dt \to \min \tag{9.4}$$

on some set Ω of admissible controls and trajectories defined for $-\infty < t < \infty$. That the lower limit of integration in (9.4) is zero, is immaterial by virtue of Theorem 9.1. Moreover, Theorem 9.1 implies that for any IOT lying in the compactum D on which the system is uniformly controllable, the limit (9.4) exists and is equal to C_0, i.e. is independent of the choice of the IOT.

We denote by Ω_n^D the set of all periodic modes $x(t)$, $u(t)$ of system (1.1) such that the cycle $x(\cdot)$ intersects the set D. The method used for Theorem 9.1 enables us to prove the following theorem.

Theorem 9.2. Let system (1.1) be uniformly controllable on the compactum $D \subset X$; $W_D(T) \neq \varnothing$ when $T > 0$. Then the quantity C_0 defined in Theorem 9.1 satisfies the equation

$$\inf_{x(\cdot),\, u(\cdot)\in\Omega_n{}^D}\ \lim_{T\to\infty}\frac{1}{T}\int_0^T F(x(t),\ u(t))\,dt = C_0$$

and, in particular, if $D = X$, then all periodic modes appear as $\Omega_n{}^D$.

This equation justifies the part played by the problem of periodic optimization as the problem of averaging. It shows that using the periodic mode it is possible to approximate by the averaged functional any optimal process of infinite duration $-\infty < t < \infty$, with a specified accuracy. If one considers that the problem of periodic optimization, which is the simplest of problems of optimal control with a functional averaged along the trajectory, which has such property, and that periodic modes are the simplest to obtain in practice, their part in the main asymptotic mode becomes clear /2–4, 13/.

10. The problem of periodic and almost-periodic optimization (PO and APO) as special cases of problems of average-optimal control. The problem of PO may be presented in three forms. The first form: determine the periodic trajectory from the set W of IOT. The second form: find the admissible periodic mode

$$\inf_{x(\cdot),\, u(\cdot)\in\Omega_n}\ \lim_{T\to\infty}\frac{1}{T}\int_0^T F(x(t),\ u(t))\,dt \tag{10.1}$$

the exact lower limit for which would be reached, and the third form; to minimize the functional

$$\frac{1}{\tau}\int_0^\tau F(x(t),\ u(t))\,dt \to \min_{u(\cdot),\, x(\cdot),\, \tau} \tag{10.2}$$

under conditions of periodicity $x(\tau) = x(0)$, where $\tau > 0$ is not specified.

Theorem 10.1. Let X be a compactum and let the system (1.1) be uniformly controllable on X. The three statements of the problem of PO are equivalent.

Proof. Implication $1 \to 2$. If $x(\cdot)$ is a periodic trajectory from W, then according to Theorem 9.1 the precise lower limit (10.1) is achieved on $x(\cdot)$, as well as on any IOT. Implication $2 \to 3$. If $x(t),\ u(t),\ -\infty < t < \infty$ are periodic functions and minimize (10.1), then according to Theorem 9.2 that minimum is equal to C_0. Let τ be the period of the process. We set $T = k\tau$ and obtain

$$\frac{1}{\tau}\int_0^\tau F(x(t),\ u(t))\,dt = \frac{1}{k\tau}\int_0^{k\tau} F(x(t),\ u(t))\,dt \to C_0 \ \text{as}\ k\to\infty$$

From this it follows that the mean value of the functional over the period for $x(\cdot), u(\cdot)$ is equal to C_0 which according to Theorem 9.2 is the exact lower bound of (10.2). The implication $3 \to 1$ was proved earlier (/2/, p.103).

Consider two forms of the statement of the problem of PPO. The first form defines the almost periodic IOT. The second form: to minimize the averaged functional (9.4) on the set of almost periodic trajectories for which the limit (9.4) exists when $T\to\infty$. No supplementary assumptions are made relative to the controls, except about measurability. If X is a compactum and the system is uniformly controllable on X, then by virtue of Theorem 9.1 it follows from the fact that $x(\cdot)$ is the solution of the problem of PPO in the first statement, if follows that $x(\cdot)$ is the solution of the problem of PPO in the second form also.

11. The problem of PPO for a linear system with a quadratic functional. Assuming for the characteristic roots λ_i of the matrix $A \in R^{n\times n}$

$$\operatorname{Re}\lambda_i < 0,\ 1 \leqslant i \leqslant n \tag{11.1}$$

we shall consider the linear system of control

$$\frac{dx}{dt} = Ax + Bu,\quad x\in R^n,\ u\in R^r \tag{11.2}$$

We denote by $L_n{}^2$ the set of periodic vector functions $u(t)\in R^r$ with all possible periods that are summable together with the scalar product of $(u(t),\ u(t))$ on any compactum from $(-\infty, \infty)$. According to (11.1) a single periodic trajectory (11.2) corresponds to each function $u(\cdot)\in L_n{}^2$. We denote by $\Omega_n{}^2$ the set of periodic pairs $x(\cdot),\ u(\cdot)\in L_n{}^2$, and by Ω_g the subset of $\Omega_n{}^2$ consisting of sinusoidal or constant functions, i.e. if $x(\cdot),\ u(\cdot)\in\Omega_g$, then all components $x(t),\ u(t)$ are sinusoidal of equal frequencies, or constant.

Consider the sinusoidal control $u_\omega(t) = [u_1\sin(\omega t+\psi_1),\ldots, u_r\sin(\omega t+\psi_r)]^*$ as $\omega\to\infty$. Then the sinusoidal trajectory which corresponds to it converges uniformly $x_\omega(t)\to 0$ with respect to t in accordance with (11.1). Hence it is possible to give meaning to the consideration of the pair $x_\infty(\cdot), u_\infty(\cdot)$ of sinusoidal trajectories and controls of infinite frequency, assuming $x_\infty(t) \equiv 0$, and as $u_\infty(\cdot)$ considering $u_\omega(t)$ as $\omega \to \infty$. We denote the set of these pairs of infinite frequency by $\Omega_{\omega=\infty}$.

We now have the problem of maximizing the averaged functional

$$P = \lim_{T\to\infty} \frac{1}{T} \int_0^T [(x, Dx) + (u, Gu)]\, dt \to \max \tag{11.3}$$

without any assumptions as to the matrices D and G. Hence the maximization can be replaced by minimization. The parentheses $(\cdot, \cdot)$ denote here a scalar product. As the set of pairs of Ω on which we seek (11.3), we take the subset of all pairs composed from the sums

$$u(t) = \sum_{i=0}^{\infty} u^i(t); \quad x(t) = \sum_{i=0}^{\infty} x^i(t); \quad u^i(\cdot), \ x^i(\cdot) \in \Omega_n^2 \cup \Omega_{\omega=\infty} \tag{11.4}$$

that satisfy the averaged limit on the control on each entry for given α_k

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u_k^2(t)\, dt \leqslant \frac{\alpha_k^2}{2}, \quad 1 \leqslant k \leqslant r \tag{11.5}$$

12. Contraction of the set of admissible pairs.

We contract (11.4) to

$$u(t) = \sum_{i=0}^{r-1} u^i(t), \quad x(t) = \sum_{i=0}^{r-1} x^i(t); \quad u^i(\cdot), \ x^i(\cdot) \in \Omega_g \cup \Omega_{\omega=\infty}. \tag{12.1}$$

Theorem 12.1. The exact upper limits (11.3) in problems (11.1)–(11.5) and (11.1), (11.2), (11.5) and (1.2.1) are the same.

Proof. Consider the control

$$u_k(t, N) = \frac{U_k^\circ}{\sqrt{2}} + \sum_{i=1}^{N-1} U_k^i \sin(\omega_i t + \psi_{ik}); \quad 1 \leqslant k \leqslant r; \quad \omega_j \neq \omega_i \ \text{ when } \ j \neq i \tag{12.2}$$

to which corresponds the stable solution (11.2) of the form

$$x_p(t, N) = \sum_{i=0}^{N-1} x_p^i; \quad x_p^\circ = \frac{X_p^\circ}{\sqrt{2}}; \quad x_p^i = X_p^i \sin(\omega_i t + \varphi_{ip}); \quad 1 \leqslant p \leqslant n \tag{12.3}$$

It is assumed that $\omega_{N-1} = \infty$. Then $X_p^{N-1} = 0$ for $1 \leqslant p \leqslant n$.

Substitution of (12.2) into (11.5) yields

$$\sum_{i=0}^{N-1} (U_k^i)^2 \leqslant \alpha_k^2, \quad 1 \leqslant k \leqslant r \tag{12.4}$$

We put

$$[U^i]^2 = \mathrm{col}\,[(U_1^i)^2, \ldots, (U_r^i)^2], \quad P_u^i = \frac{1}{2} \sum_{k=1}^{r} (U_k^i)^2 \tag{12.5}$$

$$P_x^i = \lim_{T\to\infty} \frac{1}{T} \int_0^T [(x^i(t), Dx^i(t)) + (u^i(t), Gu^i(t))]\, dt$$

Then because $\omega_i \neq \omega_j$ for $i \neq j$ we have for (12.2) and (12.3) the optimal criterion (11.3) in the form

$$P(x(\cdot, N), u(\cdot, N)) = \sum_{i=0}^{N-1} P_x^i \tag{12.6}$$

Considering the relations (12.4) and (12.5), to prove the theorem it is sufficient to show that instead of (12.2), it is possible to select a control of the form

$$u_k(t, r) = \frac{C_k^\circ}{\sqrt{2}} + \sum_{i=1}^{r} C_k^i \sin(\omega_i t + \psi_{ik}), \quad 1 \leqslant k \leqslant r \tag{12.7}$$

$$\omega_i \neq \omega_j \quad \text{when} \quad i \neq j, \ P_u^\circ P_u^r = 0$$

(i.e. where $[U^\circ]^2 = 0$ or $[U^r]^2 = 0$) such that for $x(t, r)$ that corresponds to (12.7), the inequality

$$P(x(\cdot, N), u(\cdot, N)) \leqslant P(x(\cdot, r), u(\cdot, r)), \ \sum_{i=0}^{N-1} [U^i]^2 = \sum_{i=0}^{r} [C^i]^2 \tag{12.8}$$

is correct since the functions $u(\cdot) \in L_n^2$ can be expanded in a Fourier series, and condition (12.8) implies the possibility of reducing the number of harmonics, including the constant, to the number of inputs of system (11.2) without violating constraints and without diminishing the criterion (11.3).

The vectors $[U^i]^2, 0 \leqslant i \leqslant N-1$ are linearly independent for $N > r$. Hence we can find simultaneously non-zero $\beta_0, \ldots, \beta_{N-1}$ such that

$$\beta_0 [U^i]^2 + \ldots + \beta_{N-1} [U^i]^2 = 0 \tag{12.9}$$

It suffices to prove that for $N > r$ we can pass from N harmonics to $N-1$ so as to have

$$P(x(\cdot, N), u(\cdot, N)) \leqslant P(x(\cdot, N-1), u(\cdot, N-1)) \tag{12.10}$$

$$\sum_{i=0}^{N-1} [U^i]^2 = \sum_{i=0}^{N-2} [C^i]^2$$

Then inequality (12.8) may be obtained from (12.10) by induction.

If $[U^i]^2 = 0$, inequality (12.10) is proved. We assume that $[U^i]^2 \neq 0$ when $0 \leqslant i \leqslant N-1$. Then by virtue of (12.9) among β_i we have positive and negative quantities. To be specific we assume $\beta_0, \ldots, \beta_\nu \geqslant 0$, $\beta_{\nu+1}, \ldots, \beta_{N-1} < 0$. Adding r scalar equations (12.9), we obtain $\beta_0 P_u^\circ + \ldots + \beta_{N-1} P_u^{N-1} = 0$. Let us calculate two coefficients

$$k_l = \frac{\beta_0 P_x^\circ + \ldots + \beta_\nu P_x^\nu}{\beta_0 P_u^\circ + \ldots + \beta_\nu P_u^\nu}$$

$$k_s = \frac{(-\beta_{\nu+1}) P_x^{\nu+1} + \ldots + (-\beta_{N-1}) P_x^{N-1}}{(-\beta_{\nu+1}) P_u^{\nu+1} + \ldots + (-\beta_{N-1}) P_u^{N-1}}$$

and assume to be specific that $k_l \leqslant k_s$. We select $\beta = \max \beta_j$ from $0 \leqslant j \leqslant \nu$. Assuming that $\beta = \beta_1$

$$\frac{\beta_0}{\beta_1} P_u^\circ + P_u^1 + \frac{\beta_2}{\beta_1} P_u^2 + \ldots + \frac{\beta_\nu}{\beta_1} P_u^\nu = \left(-\frac{\beta_{\nu+1}}{\beta_1}\right) P_u^{\nu+1} + \ldots + \left(-\frac{\beta_N}{\beta_{N-1}}\right) P_u^{N-1}$$

Using the formulae

$$C_k^i = \sqrt{1 - \frac{\beta_i}{\beta_1}}\, U_k^i, \quad 0 \leqslant i \leqslant N-1, \quad 1 \leqslant k \leqslant r$$

we change the amplitudes of the harmonics u. Then

$$\bar{P}_u^i = \left(1 - \frac{\beta_i}{\beta_1}\right) P_u^i, \quad \bar{P}_x^i = \left(1 - \frac{\beta_i}{\beta_1}\right) P_x^i, \quad [C^1] = 0$$

i.e. the number of harmonics is reduced by one and

$$\sum_{i=0}^{N-1} [C^i]^2 = \sum_{i=0}^{N-1} \left(1 - \frac{\beta_i}{\beta_1}\right) [U_k^i]^2 = \sum_{i=0}^{N-1} [U_k^i]^2 - \frac{1}{\beta_1} \sum_{i=0}^{N-1} \beta_i [U_k^i]^2 = \sum_{i=0}^{N-1} [U_k^i]^2$$

From the inequality $k_l \leqslant k_s$, taking into account the equality of the denominators in k_l, k_s, we obtain

$$\beta_0 P_x^\circ + \ldots + \beta_\nu P_x^\nu \leqslant -\beta_{\nu+1} P_x^{\nu+1} - \ldots - \beta_{N-1} P_x^{N-1}$$

Then the following criterion corresponds to the new control amplitudes:

$$P(\bar{x}(\cdot), \bar{u}(\cdot)) = \sum_{i=0}^{N-1} \bar{P}_x^i = \sum_{i=0}^{N-1} \left(1 - \frac{\beta_i}{\beta_1}\right) P_x^i = P(x(\cdot, N), u(\cdot, N)) -$$

$$\frac{1}{\beta_1} \left[\sum_{i=0}^{N-1} \beta_i P_x^i\right] \geqslant P(x(\cdot, N), u(\cdot, N))$$

which is identical with (12.10) apart from the numbering of the harmonics.

The averaged functional (11.3) of the form

$$I = I(U^\circ, \ldots, U^r, \omega_1, \ldots, \omega_r, \psi_1, \ldots, \psi_r) \to \min \tag{12.11}$$

$$U^i = \operatorname{col}[U_1^i, \ldots, U_r^i], \quad \psi_i = \operatorname{col}[\psi_{i1}, \ldots, \psi_{ir}]$$

corresponds to control in the form of the sum of harmonics (12.7). Here $U^\circ = 0$ or $U^r = 0$, i.e. the over-all number of harmonics, including the constant component, does not exceed r, and the constraints (11.5) have the form

$$\sum_{i=0}^{r} (U_k^{i})^2 \leqslant \alpha_k^2, \quad 1 \leqslant k \leqslant r \tag{12.12}$$

According to (12.12) and (12.7) the regions of variation of U^i, ψ_i are compact. All values from 0 to ∞ are admissible for ω_i. Hence condition (11.1) enables us to state the following theorem.

Theorem 12.2. The problem of non-linear programming (12.11), (12.12) has a solution which determines the solution of problem (11.2), (11.3), (11.5), (12.1) in the form (12.7), (12.1).

Corollary 12.1. The problem (11.1)–(11.5) has a solution that is provided by the solution of problem (12.11), (12.12) in the form (12.7), (12.1).

This shows that when the number of inputs $r \geqslant 2$ and the frequencies $\omega_1, \ldots, \omega_{r-1}$ obtained in the solution of problem (12.11), (12.12) are incommensurable, the solution is obtained in the class of almost periodic functions.

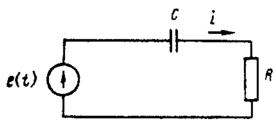

Fig.1

Remark. Problem (11.1)–(11.5) may be treated as one of maximum power transmission to the load with power constraint on each input. Besides it is seen that the Theorems 12.1, 12.2 and Corollary 12.1 hold also when functional (11.3) is replaced by the functional

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} [(x, Dx) + (x, Lu) + (u, Gu)]\, dt \to \max$$

where L is the matrix $n \times r$.

Example. Consider the problem of supplying maximum power to the resistance R in the electric circuit shown in Fig.1, with

$$P = \lim_{T\to\infty} \frac{1}{T} \int_0^T i^2 R\, dt \to \max$$

and a constraint on the control provided by the electromotive force $e(t)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T e^2(t)\, dt \leqslant \frac{\alpha^2}{2}, \quad \alpha > 0$$

We denote the voltage across the capacitor by x, the current by i, the capacitance by C, and we obtain Kirchhoff's second law $x + iR = e$. For the capacitance we have

$$\frac{dx}{dt} = \frac{1}{C} i$$

from which follows the differential equation

$$\frac{dx}{dt} = (e - x)(RC)^{-1}$$

According to Corollary 12.1 the solution is provided by a single harmonic $e = \alpha \sin \omega t$, from which $i = \alpha (R^2 + 1/(\omega C)^2)^{-1/2} \sin(\omega t + \varphi)$. It is seen that the maximum of $P = \alpha^2/(2R)$ is reached when $\omega = \infty$. This corresponds to the fact that the maximum transmission of power to the load, for the chain considered here, corresponds to frequencies as high as desired. Mathematically, this means that $e(t) = \alpha \sin \omega t$ is considered as the solution when $\omega \to \infty$. The same problem for a chain differing from the one in Fig.1 by the addition of an inductance L has a solution $e(t) = \alpha \sin \omega t$, $\omega = (LC)^{-1/2}$ with the same maximum power.

REFERENCES

1. PANASYUK A.I. and PANASYUK V.I., On the asymptotic form of the trajectories of a class of problems of optimization. Avtomatika i Telemekhanika, No.8, 1975.
2. PANASYUK A.I. and PANASYUK V.I., The Asymptotic Optimization of Non-linear Control Systems. Minsk, Izd. Belorusskogo Univ. 1977.
3. PANASYUK A.I. and PANASYUK V.I., On the behaviour of infinite optimal trajectories of a class of continuous dynamic systems. Differents. Uravneniya, 6, 1982.
4. PANASJUK A. and PANASJUK V., Die Wichtigsten Leitsätze der magistralen asymptotischen Theorie der optimalen Steuerung. In: 27 Intern. Wiss. Kolloq., Ilmenau 1982. 5, Vortraqsz., 1, 2, Ilmenau.
5. ROMANOVSKII I.V., Algorithms for Solving Extremal Problems. Moscow, Nauka, 1977.
6. GUSEV D.E. and YAKUBOVICH V.A., The main line theorem in the problem of continuous optimization. Vestn. LGU. Ser. Matem., Mekhan., Astron., 1, 1983.

7. DUKEL'SKII M.S. and TSIRLIN A.M., Conditions of unsteadiness of the optimal steady mode of a controlled object. Avtomatika i Telemekhanika 9, 1977.
8. CHERNOUS'KO F.L., AKULENKO L.D. and SOKOLOV B.N., Control of Oscillations, Moscow, Nauka, 1980.
9. ANISOVICH V.V. and KRYUKOV B.I., On the optimization of almost-periodic oscillations. Avtomatika i Telemekhanika, 12, 1981.
10. VASIL'EVA A.B. and DIMITRIEV M.G., Singular perturbations in problems of optimal control. In: Progress of Science and Technology. Mathematical Analysis. 20, Moscow, VINITI, 1982.
11. PLOTNIKOV V.A., The asymptotic investigation of equations of controlled motion. Izv. AN SSSR, Tekhnicheskaya Kibernetika, 4, 1984.
12. PANASYUK V.I., The problems of main trajectories in discrete problems of optimal control. Avtomatika i Telemekhanika, 8, 1981.
13. PANASYUK V.I., Main periodic trajectories in discrete problems of optimal control. Avtomatika i Telemekhanika, 9, 1983.
14. FLEMING W. and RISCHEL R., Optimal Control of Determinate and Stochastic Systems. Moscow, Mir, 1978.
15. KELLY J.L., General Topology. Moscow, Nauka, 1981.

Translated by J.J.D.

PMM U.S.S.R.,Vol.49,No.4,pp.413-422,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

A GAME OF OPTIMAL PURSUIT OF ONE NON-INERTIAL OBJECT BY TWO INERTIAL OBJECTS*

A.YU. LEVCHENKOV and A.G. PASHKOV

A game in which one controlled object is pursued by two others is studied. The pursuing objects are inertial, and the pursued object is not. The duration of the game is fixed. The payoff functional is the distance between the pursued object and the closest pursuer at the instant when the game ends. An algorithm for determining the payoff function for all possible positions is constructed. It is shown that the game space consists of several domains in which the payoff is expressed analytically, or is determined by solving a certain non-linear equation. Strategies of the pursuers which guarantees them a result as close to the game payoff as desired are indicated.

The optimal solution of a game of pursuit when one inertial object pursues a non-inertial one was obtained earlier in /1/. The present paper is related to the investigations reported in /1—10/.

1. Let the motions of the pursuers $P_i\,(x^i)\;(i=1,2)$ and of the pursued object $E\,(z)$ be described by the equations

$$x_1^{\cdot i}=x_3^{\,i},\quad x_3^{\cdot i}=u_1^{\,i},\quad x_2^{\cdot i}=x_4^{\,i},\quad x_4^{\cdot i}=u_2^{i},\quad z_1^{\cdot}=v_1,\quad z_2^{\cdot}=v_2 \tag{1.1}$$

The control vectors of the pursuers and the pursued satisfy the constraints

$$((u_1^{\,i})^2+(u_2^{\,i})^2)^{1/2}\leqslant\mu>0,\quad (v_1^2+v_2^2)\leqslant\nu \tag{1.2}$$

The game is studied over the time interval $[t_0,\vartheta)$. The payoff functional is the distance between the pursued object and the nearest pursuer at the instant $t=\vartheta$ that the game ends, i.e.

$$\gamma=\min_i\,[(z_1(\vartheta)-x_1^{\,i}(\vartheta))^2+(z_2(\vartheta)-x_2^{\,i}(\vartheta))^2]^{1/2} \tag{1.3}$$

As a result of the change of variables $y_j^{\,i}=x_j^{\,i}+(\vartheta-t)\;x_{j-2}^{i}\;(j=1,2)$, which means passing to considering the centres of regions of attainability of the inertial objects, relations (1.1)—(1.3) take the form

$$y_j^{\cdot i}=(\vartheta-t)\,u_j^{\,i},\quad y_j^{\,i}(t_0)=x_j^{\,i}(t_0)+(\vartheta-t_0)\,x_{j-2}^{i}(t_0) \tag{1.4}$$

$$\gamma=\min_i\,[(z_1(\vartheta)-y_1^{\,i}(\vartheta))^2+(z_2(\vartheta)-y_2^{\,i}(\vartheta))^2]^{1/2} \tag{1.5}$$

At the instant $t=\vartheta$ the values of γ found from (1.3) and (1.5) are identically equal.

We denote the centres of the attainability regions by P_i. For the positions where $P_1^c=P_2^c$, the payoff of the two-to-one game, denoted by ρ^{21}, is identical with the payoff of the one-to-one game denoted by ρ^{11}. Henceforth we consider those initial positions for which $P_1^\circ\neq P_2^\circ$.

---

*Prikl.Matem.Mekhan.,49,4,536-547,1985